CSC 2420 Toronto
UW CSE 521/522
Cornell CSC 6820
Caltech CS 139
CMU 451/651

Outline

Part I. Classic Online Algorithms.

- Standard definitions (CR. class of adversaries)
- online problems.
  - ski-rental
  - paging
  - one-way trading
  - online knapsack
  - online bin packing
  - bit guessing
  - ⋮

3-4 weeks

- case studies of online algorithms & randomized algs.

- Prepare you w/ the basic knowledge & analysis tools.

– New Year Holidays.

Part II. Online learning. covers a wide range of algs

4 weeks

- online optimization (basic problem formulation, Regret)
- MWA and its applications
- adaptive regret
- application in games.
- Application in communication problems
- Bandit problems. (EXP3)

Part III Recent Topics

- online decision-making w/ dynamics
- online algorithms w/ predictions

2-3 weeks
  - Ski-rental w/ ML predictions (revisit)

weeks
  - LQC w/ predictions
  - MPC properties
  - other recent advances

- Guest lectures / presentations

3 weeks

Topis TBP.

Lecture 01.

- What are online problems.
    - ski-rental
    - paging

Lecture 02

- classic online algorithms and analysis PART I

Deterministic

- Bin Packing (NP-hard)

Problem description (Classic Bin Packing)

Given a sequence of items $(x_1, \dots, x_n)$ $(x_i \in [0,1])$.
weight

Goal: Minimize the # of bins needed to pack all items.

The size of each bin is 1.

Input: $(x_1, x_2, \dots, x_n)$

Output: $6: \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$ for some $m \in N_+$.

objective: $\min_{6} m$ ——— (1)

$$\text{s.t.} \sum_{j: 6(j)=i} x_j \leq 1. \quad \forall i \in [m].$$

Now, consider three online algorithms:

Next fit: If the current bin is full, open a new bin and fit

First fit: Upon the arrival of a new item, find the first bin among all opened feasible bins and fit the item. if there exists such a bin; otherwise, open a new bin. (Next page pseudo code)

Best fit: upon the arrival of a new item, find the bin with the smallest remaining space among all opened feasible bins, and fit the item. if such a bin exists; otherwise, open a new bin.

Intuitively, Best fit / First fit may outperform Next fit. by how much?

How to formally analyze this?

- we'd like to optimize $m(ALG)$, which depends on n.
- what if we compute $\dfrac{m(ALG)}{m(OPT)}$? offline optimization

$m(OPT) \to$ optimal sol of (1).

Definition 2.1. (competitive ratio)

Let $C(ALG:\theta)$ be the cost of an online algorithm ALG, parameterized by θ for a minimization problem. Let $c(OPT:\theta)$ be the optimal cost. The competitive ratio (CR) of ALG, denoted by $CR(ALG)$ is defined as

$$CR(ALG) := \limsup_{c(OPT) \to \infty} \sup_{\theta} \frac{c(ALG:\theta)}{c(OPT:\theta)}$$

Example. First ~~fit~~

procedure First Fit.

$m \leftarrow 0$ (# of opened bins)

$R \leftarrow \{\}$ a dictionary tracking remaining spaces in opened bins.

while $j \leq n$. do

flag $\leftarrow$ False

for $i = 1$ to m. do

if $x_j < R[i]$. then

$R[i] \leftarrow R[i] - x_j$

$6(j) \leftarrow i$

Flag $\leftarrow$ True

break

if flag = False. then # open a new bin.

$m \leftarrow m + 1$

$R[m] \leftarrow 1 - x_j$

$6(j) \leftarrow m$

$j \leftarrow j + 1$.

Remark.

- Can be defined similarly for maximization problems.
- In our last lecture, we've seen a few examples.
- sup is over problem instances, depending on concrete contexts.

Theorem 2.1

$$\limsup_{n\to\infty} \sup_{\{x_1,\dots,x_n\}} \frac{m(ALG)}{m(OPT)}$$

$$CR(\text{Next fit}) \leq 2.$$

How much weight has been occupied ?

proof: Define $B[i] = 1 - R[i]$, $\forall i$.

WLOG, assume $m(\text{Next fit})$ is even.

Then $B[1] + B[2] > 1$. (why ?)

by the alg. at least one item in the 2nd bin cannot be placed in the 1st bin.

Similarly.

$B[2i-1] + B[2i] > 1$.

$\forall i \in \{1, \dots \frac{m}{2}\}$ m is even

Summing up. $\underbrace{\sum_{i=1}^{\frac{m}{2}} B[2i-1] + B[2i]}_{\text{what is this ?}} > \frac{m}{2}$

$$\overset{\shortparallel}{\sum_{j}^{n} w_j} \leq m(OPT)$$

$\Rightarrow m \leq 2 \cdot m(OPT)$, $\forall n \,\&\, \{x_1 \dots x_n\}$.

$\Rightarrow CR(\text{Next fit}) \leq 2$. #.

Q: Can we find an instance of $\{x_1 \dots x_n\}$ and find a lower bound on $CR(\text{Next fit})$?

Theorem 2.2

$$CR(\text{Next fit}) \geq 2.$$

Corollary 2.1

$$CR(\text{Next fit}) = 2.$$

Proof: Fix arbitrarily small $\varepsilon > 0$ such that $n := \frac{1}{\varepsilon} \in \mathbb{N}$.

Consider the following input:

$$\underbrace{(1-\varepsilon, 2\varepsilon)\ (1-\varepsilon, 2\varepsilon), \cdots (1-\varepsilon, 2\varepsilon)}_{2n}, \underbrace{\varepsilon, \cdots, \varepsilon}_{n}$$

How many bins are required by Next fit?

$2n$, since the first pattern will force the alg to use a new bin for every new item. $(1-\varepsilon+2\varepsilon > 1)$

Now, what is $m(OPT)$?

$$m(OPT) \le n + \underbrace{2\varepsilon \cdot n}_{\substack{\text{the last } n \text{ items} \\ \text{can be put into 2 bins}}} = n + 2\varepsilon \cdot \frac{1}{\varepsilon} = n+2$$

$$\Rightarrow CR(\text{Next fit}) \ge \frac{2n}{n+2} \to 2, \text{ when } n \to \infty \quad \#.$$

Remark. This weighting technique can be made more general for analyzing other online bin packing algorithms.

How to analyze First fit and Best fit?

Idea: Define a weight function $w: [0,1] \to \mathbb{R}$ that satisfies

- P1 • $w(x) \ge x$

Define $w(S) := \sum_{i \in S} w(x_i)$ as an extension (set of indices of items in bin i by ALG)

- P2 • $\exists$ a constant $k_0 \in \mathbb{N}$ and $(\beta_j : j \in [m])$ s.t.
$$w(S_j) \ge 1 - \beta_j, \quad \sum_{j=1}^{m} \beta_j \le k_0, \quad \forall j \in [m].$$
- P3 • $\forall k \in \mathbb{N}$, $y_1, \ldots, y_k \in [0,1]$, we have $\sum_{i=1}^{k} y_i \le 1$ implies $\sum_{i=1}^{k} w(y_i) \le \gamma \to$ a parameter

Assume such w exists. for given ALG and $\gamma > 0$.

we have

$CR(ALG) \leq \gamma$, since by P2.

$$w([n]) = \sum_{i=1}^{n} w(x_i) = \sum_{i=1}^{m} w(S_i) \geq \sum_{i=1}^{\overset{m(ALG)}{m}} (1-\beta_i) = m - \sum_{i=1}^{m} \beta_i > m - k_0 \quad (1)$$

Denote by Q_i the set of indices of items placed in bin i by OPT and write $m(OPT) =: m^*$ for simplicity.

$$w([n]) = \sum_{i=1}^{m^*} w(Q_i) \leq \gamma m^* \quad (2)$$

$(1)+(2) \Rightarrow \gamma m^* \geq m - k_0$

$\Rightarrow CR(ALG) \leq \gamma$.

Using this technique. we're able to show:

Theorem 2.3.

$$CR(\text{First fit}) \leq \frac{17}{10}$$

$$CR(\text{Best fit}) \leq \frac{17}{10}.$$

proof sketch:

consider

$$w(x) = \begin{cases} \frac{6}{5}x, & 0 \leq x \leq \frac{1}{6} \\ \frac{9}{5}x - \frac{1}{10}, & \frac{1}{6} \leq x \leq \frac{1}{3} \\ \frac{6}{5}x + \frac{1}{10}, & \frac{1}{3} < x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1. \end{cases}$$

check w satisfies P3 s.t.

if $\sum_{i=1}^{k} y_i \leq 1$, then $\sum_{i=1}^{k} w(y_i) \leq 1.7$

$\forall k \in \mathbb{N}$ and $y_1 \ldots y_k \in [0,1]$. and P2. #

• References.

Johnson: et. al.

Yao. $\frac{5}{3}$ refined first fit

current best ratio 1.57829 Balogh et. al

Best lower bdd 1.54278. Balogh et. al